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Let the albedo of single scattering λ in an isotropically scattering homogeneous optically semi-infinite atmosphere be complex

$$\lambda = \lambda_1 + i\lambda_2. \tag{1}$$

We assume that in this case the source function B for a large range of radiative transfer problems in given type of atmospheres can still be described by the well-known Fredholm integral equation

$$B(\tau) = \frac{1}{2}\lambda \int_0^\infty E_1(t-\tau)B(t)dt + B_0(\tau), \tag{2}$$

where τ is the optical depth and the exponential integral is expressed in the form

$$E_n(x) = \int_0^1 \exp(-|x|/s)s^{n-2}ds \tag{3}$$

$$B(\tau) = B_0(\tau) + \int_0^\tau \Phi(t) B_0(t) dt. \tag{4}$$

According to Sobolev [17] the solution of Eq.(2) in the case of the Milne problem is

$$B(\tau) = \exp(\kappa\tau) + \int_0^\tau \exp(\kappa(\tau - t)\Phi(t) dt, \tag{5}$$

where κ is the smallest positive zero of the characteristic equation in the real domain

$$\frac{\lambda}{2\kappa} \ln \frac{1+\kappa}{1-\kappa} = 1. \tag{6}$$

Sobolev has shown that the function Φ satisfies the following Fredholm integral equation

$$\Phi(\tau) = \frac{1}{2}\int_0^\infty E_1(|t-\tau|)\Phi(t)dt + \frac{1}{2}E_1(\tau). \tag{7}$$

This is one of the most important equations in the radiative transfer since all the relevant functions of transfer can be expressed through the resolvent function.

We try to solve Eq.(7) by approximating the kernel of it by a sum of exponents

$$E_1(\tau) = \sum_{n=1}^{N} w_n \exp(-\tau/u_n)u_n^{-1}, \tag{8}$$

$$\Phi(\tau) = \sum_{i=1}^{N} a_i \exp(-s_i \tau). \tag{9}$$

The unknown coefficients s_i are the zeros of the equation

$$1 - \lambda \sum_{n=1}^{N} \frac{w_n}{1 - s^2 u_i^2} = 0, \tag{10}$$

The approximate characteristic equation - Eq.(10) - can simply be solved when λ is real and positive since we know beforehand in which intervals to search for the zeros. This is not the case when λ is complex or negative but if we write Eq.(13) in the polynomial form

$$\sum_{i=1}^{N} c_i s_i^{2i} = 0, \tag{11}$$

where

$$c_N = \Pi_{i=1}^{N} u_i^2 \tag{12}$$

we may apply the code DZROOTS from Numerical Recipes [23].

The coefficients a_i are to be found from linear algebraic system of equations

$$\sum_{i=1}^{N} \frac{a_i}{1 - s_i u_j} = u_j^{-1}, \quad j = 1, 2, ..., N. \tag{13}$$

This system may be solved e.g. using algorithms ZGECO and ZGESL from LINPACK.

$$h(\tau,\mu) = 1 + \int_{\tau}^{\infty} \Phi(t) \exp(-(t-\tau)/\mu) dt, \tag{14}$$

$$g(\tau,\mu) = \exp(-\tau/\mu) + \int_{0}^{\tau} \Phi(t) \exp(-(\tau-t)/\mu) dt. \tag{15}$$

In our approximation these functions take on the form

$$h(\tau,\mu) = 1 + \mu \sum_{i=1}^{N} \frac{a_i \exp(-s_i\tau)}{1+s_i\mu}, \tag{16}$$

$$g(\tau,\mu) = \exp(-\tau/\mu) + \mu \sum_{i=1}^{N} \frac{a_i[\exp(-s_i\tau) - \exp(-\tau/\mu)]}{1-s_i\mu}. \tag{17}$$

4. The planetary problem

Sobolev has shown that for the planetary problem the source function takes the form [10]

$$B(\tau) = \frac{1}{4}\lambda F H(\mu_0)\Big[\exp\left(-\tau/\mu_0\right) + \int_0^{\tau}\Phi(t)\exp\left(-(\tau - t)/\mu_0\right)dt\Big], \tag{24}$$

or taking into account Eq.(15) we find that

$$B(\tau) = \frac{1}{4}\lambda F H(\mu_0) g(\tau, \mu_0). \tag{25}$$

Using our auxiliary functions these formulas can be simplified

$$I(\tau, -\mu) = \frac{1}{4}\lambda F \frac{\mu_0 H(\mu_0)}{\mu + \mu_0}[g(\tau, \mu_0) + h(\tau, \mu) - 1], \tag{28}$$

$$I(\tau, \mu) = \frac{1}{4}\lambda F \frac{\mu_0 H(\mu_0)}{\mu_0 - \mu}[g(\tau, \mu_0) - g(\tau, \mu)]. \tag{29}$$

5. The Milne problem

$$\frac{\lambda}{2\kappa} \ln \frac{1+\kappa}{1-\kappa} = 1.$$

Garcia and Siewert [22] have defined the following parameters

$$\gamma = \frac{\pi}{2}\Big(\frac{\lambda_1^2 + \lambda_2^2}{\lambda_2}\Big), \tag{30}$$

$$\varpi = \exp\left(-\lambda_1 \pi / \lambda_2\right), \tag{31}$$

and

$$\zeta = \frac{1-\varpi}{1+\varpi}. \tag{32}$$

If now

$$\gamma\zeta > 1 \tag{33}$$

the characteristic equation (6) has two zeros in the complex plane outside the slit $[-1, 1]$ but if

$$\gamma\zeta \leq 1 \tag{34}$$

there are no zeros on the complex plane.

The case with $\lambda_1 = 1$ and $\lambda_2 = 0$ - the conservative case - is well studied and the source function in that case is expressed as

$$B(\tau) = \sqrt{3}[\tau + q(\tau)], \tag{36}$$

where $q(\tau)$ is the Hopf function. In our approximation this function may be written as

$$q(\tau) = \frac{1}{\sqrt{3}}[1 + \sum_{i=2}^{N} a_i s_i^{-1}(1 - e^{-s_i\tau})]. \tag{37}$$

The Hopf function at infinity has played an important role in the radiative transfer and it is called the linear extrapolation distance. Placzek and Seidel [23] have found the formula for $q(\infty)$ in the form

$$q(\infty) = \frac{6}{\pi^2} + \frac{1}{\pi}\int_0^{\pi/2}\Big(\frac{3}{x^2} - \frac{1}{1 - x\cot x}\Big)dx. \tag{38}$$

The author has used this formula together with bc-language to find that [24]

$$q(\infty) = 0.71044608959876307273252414169915367199320133395878523909280.$$

For the conservative case the intensities are

$$I(\tau, -\mu) = \sqrt{3}[\tau + q(\tau)] + h(\tau, \mu) + 1, \tag{39}$$

$$I(\tau, \mu) = \sqrt{3}[\tau + q(\tau)] - g(\tau, \mu) \tag{40}$$

$$B(\tau) = H(\frac{1}{\kappa})e^{\kappa\tau} - h(\tau, \frac{1}{\kappa}) + 1. \tag{42}$$

In this case the intensities are

$$I(\tau, -\mu) = \frac{1}{1-\kappa\mu}[H(1/\kappa)e^{\kappa\tau} - h(\tau, 1/\kappa) + h(\tau, \mu)], \tag{43}$$

$$I(\tau, \mu) = \frac{1}{1+\kappa\mu}[H(1/\kappa)e^{\kappa\tau} - h(\tau, 1/\kappa) + 1 - g(\tau, \mu)]. \tag{44}$$

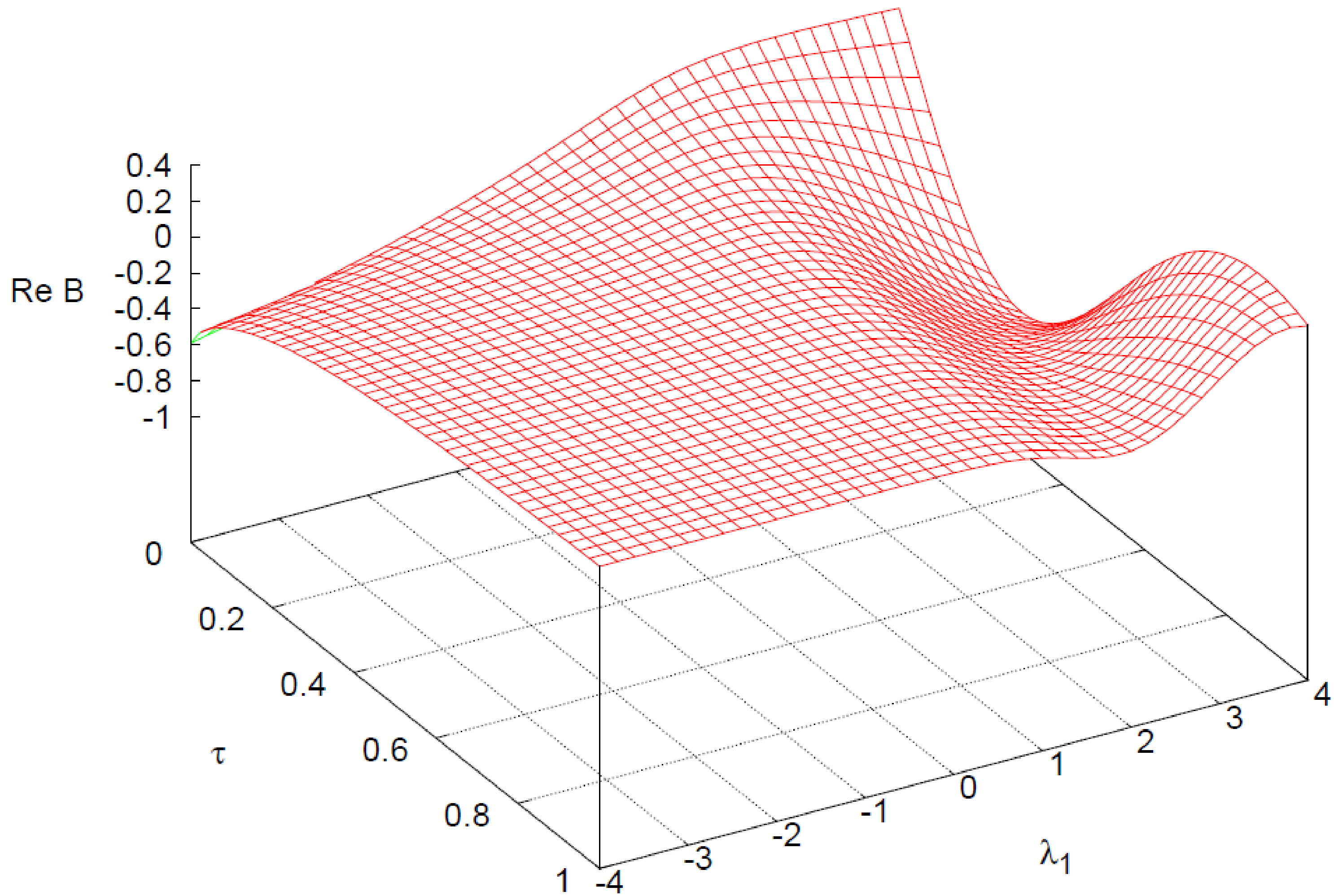

Fig.1. The real part of the source function for the planetary problem as a function of λ_1 and τ ($\mu_0 = 0.5,\ \lambda_2 = 4$).

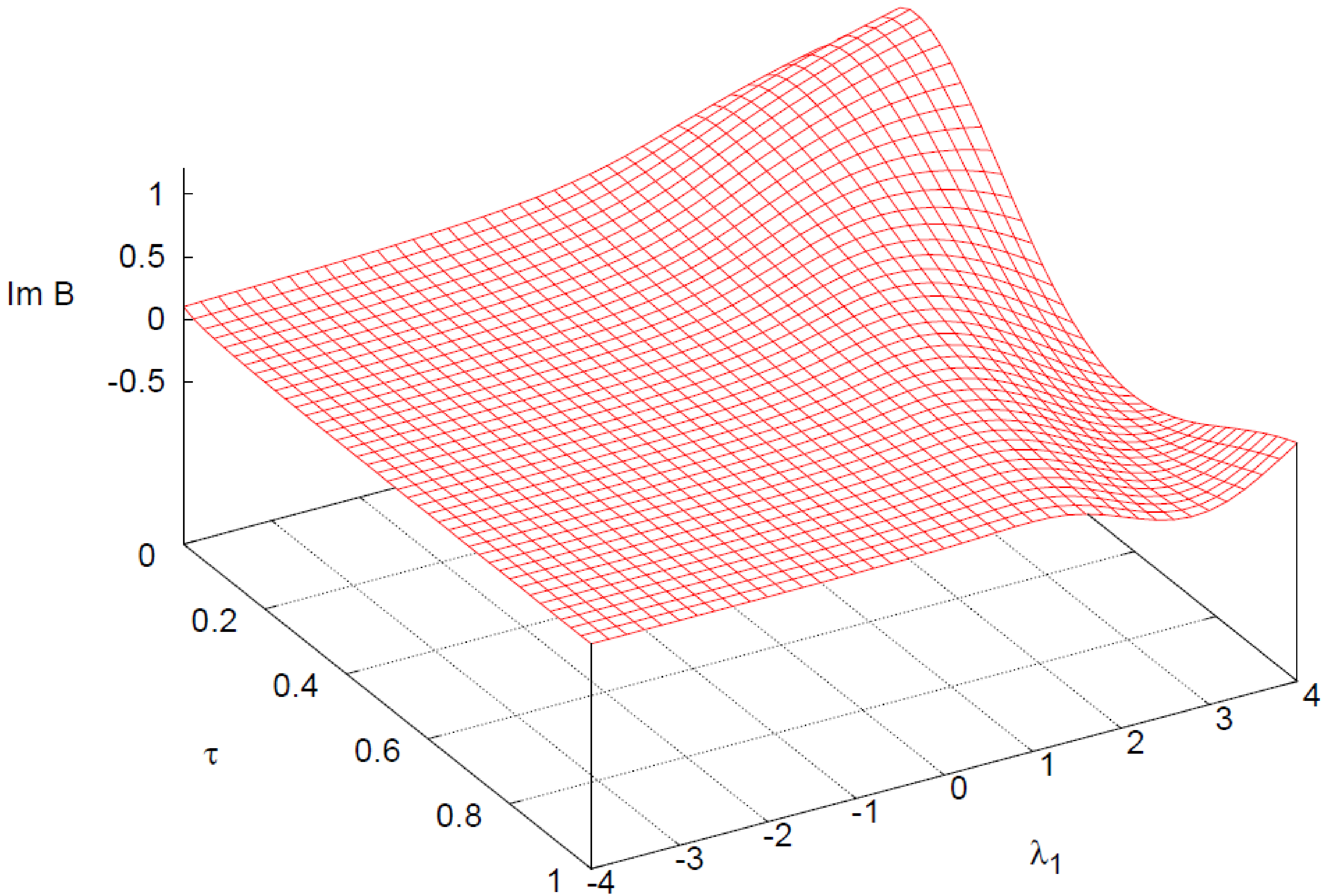

Fig.2. The imaginary part of the source function for the planetary problem as a function of λ_1 and τ ($\mu_0 = 0.5$, $\lambda_2 = 4$).

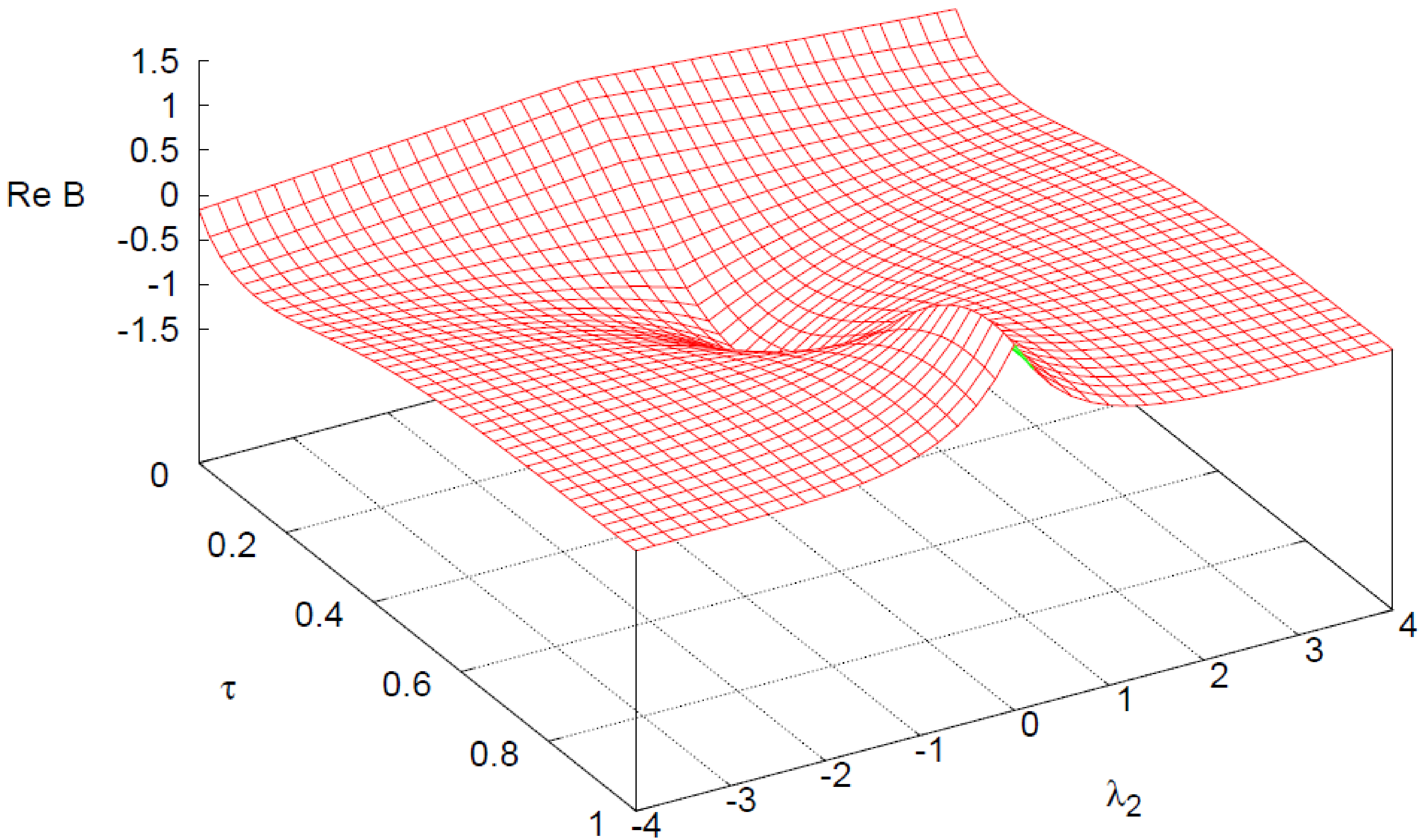

Fig.3. The real part of the source function for the planetary problem as a function of λ_2 and τ ($\mu_0 = 1.0,\ \lambda_1 = 4$).

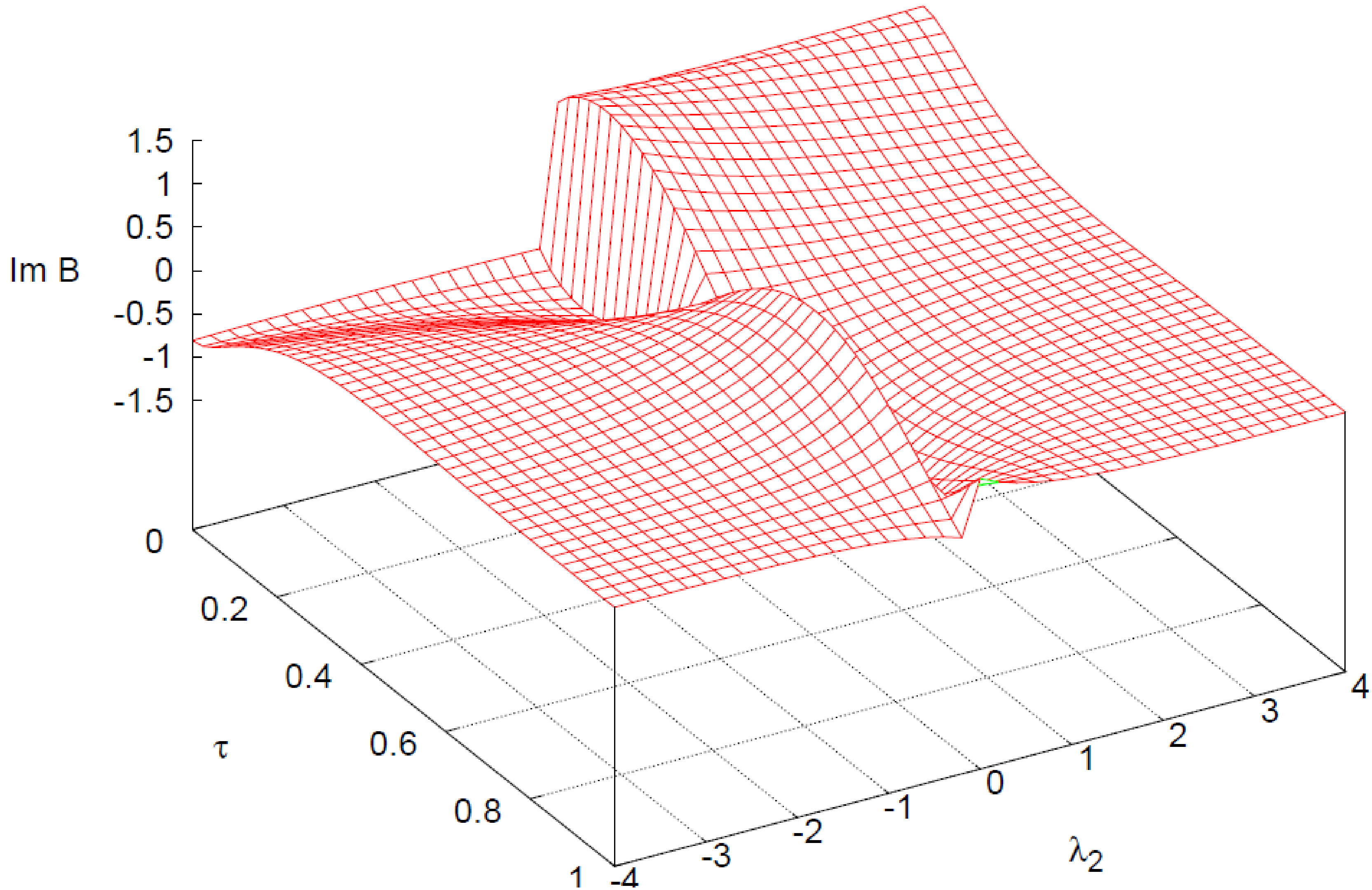

Fig.4. The imaginary part of the source function for the planetary problem as a function of λ_2 and τ ($\mu_0 = 1.0$, $\lambda_1 = 4$).

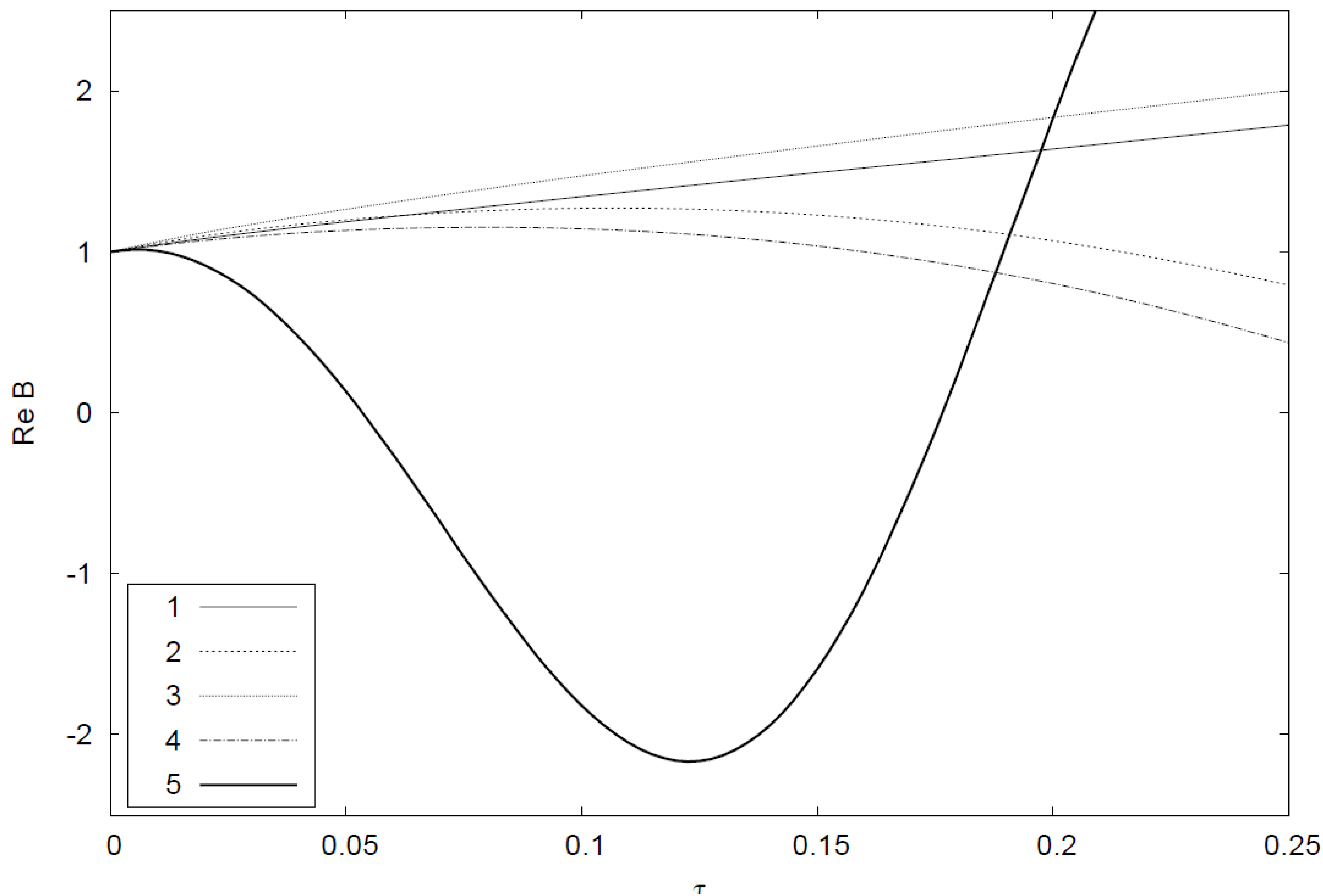

Fig.5. The real part of the source function for the Milne problem as a function of τ (1 - $\lambda = 1.2 + 0.4i$, 2 - $\lambda = 1.6 + 0.4i$, 3 - $\lambda = 2.0 + 0.4i$, 4 - $\lambda = 1.8 + 0.6i$, 5 - $\lambda = 2.4 + 0.6i$).

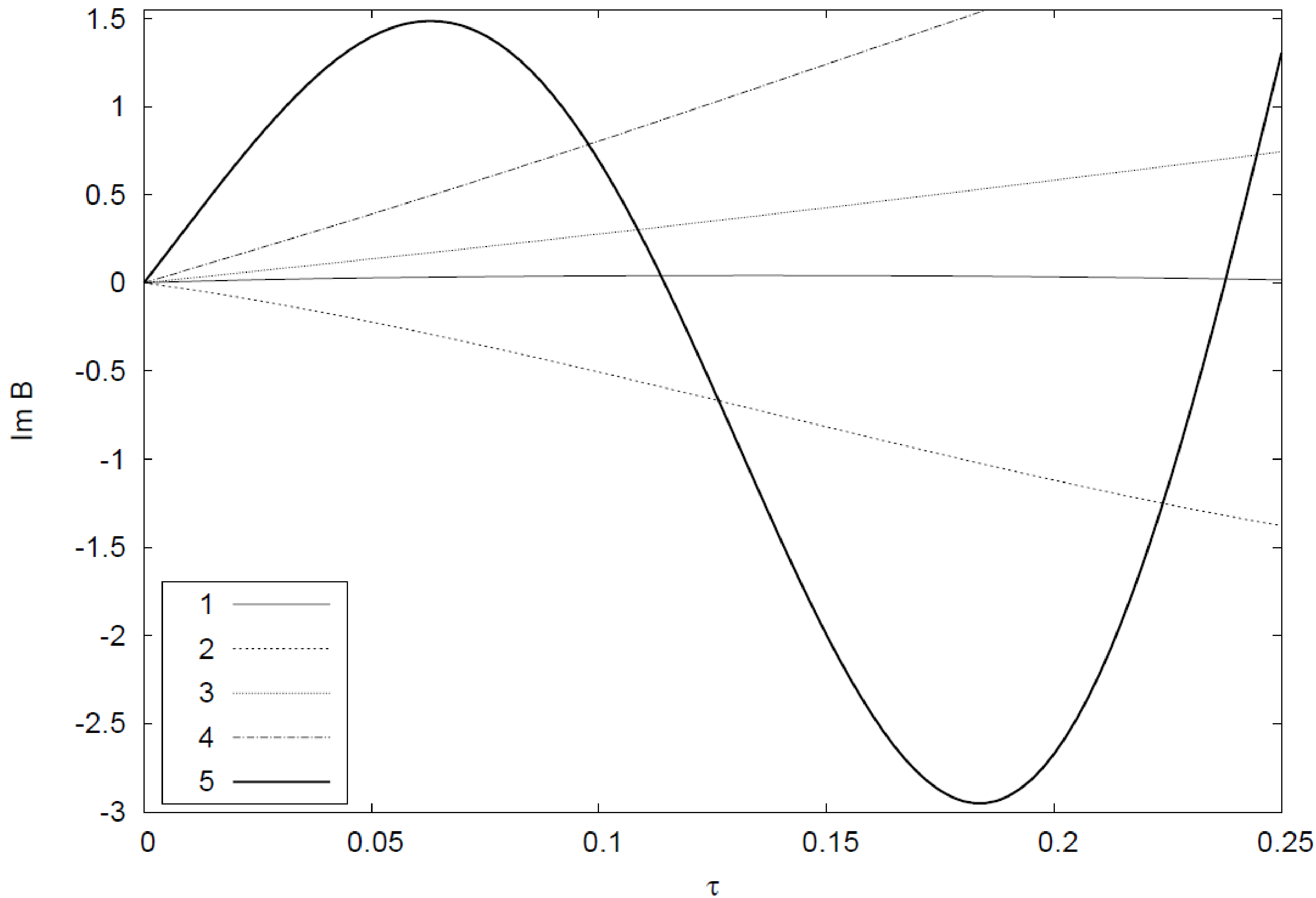

Fig.6. The imaginary part of the source function for the Milne problem as a function of τ (the key is the same as in Fig.3).

Tänu vastupidavuse eest!